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ORIGINAL PAPER

The braid index of polyhedral links

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Abstract Polyhedral links have been used to model DNA polyhedra and protein catenanes. Some topological characteristics of a type of polyhedral links fabricated from a polyheron by the method of 'n-branched curve and X-tangled covering' have been elucidated. In this paper, we pay close attention to their braid index considered significant in view of DNA nanotechnology, and proved that the MFW inequality is sharp for the polyhedral links. Our results demonstrate that the braid index of the links is directed by their crossing numbers. In addition, the studies of the polyhedral links can facilitate the research of the properties of DNA molecules, and can characterize their structural complexity.

Keywords DNA polyheron · Polyhedral links · MFW inequality · Braid index · Complexity

1 Introduction

Braid index, a fundamental topological invariant, is a main issue in the field of mathematics and chemistry. In chemistry, the braid index is sometimes used to describe the complexity of a molecule [1], and it can also provide the most structural information. The braid index of knotted hydrocarbon complexes can facilitate the study of their properties [2].

In recent years, however, a variety of interlinked polyhedra with more exotic topologies have been synthesized. Using DNA molecules, experimental scientists have chemically realized many DNA polyhedral nanostructure including DNA Platonic

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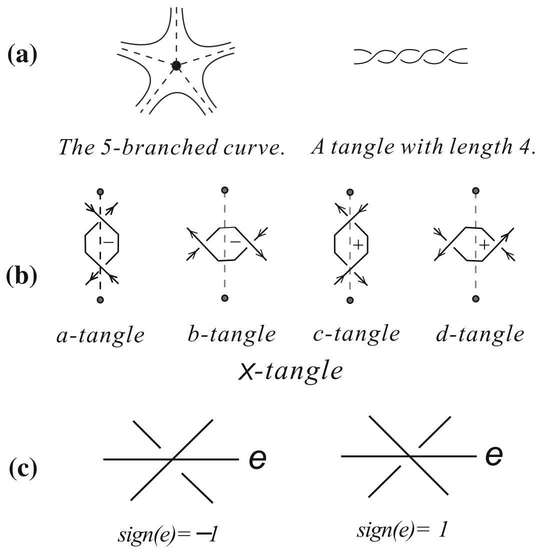


Fig. 1 **a** *5-branched curve*, and *a tangle with length 4*; **b** X-tangle, which including *a-tangle*, *b-tangle*, *c-tangle*, *d-tangle*; **c** The sign of edge e

polyhedra and DNA Archimedean polyhedra [3–10]. These peculiar objects provide some topological nontrivial structures embedded in 3D space. In the face of such topologically complex molecules, the description and quantification of their construction inspire us great chemical interest. Here, the braid index as a famous index of DNA entanglement complexity is introduced to characterize their structural complexity.

Inspired by the novel DNA polyhedra, Zhang and Qiu et al fabricated many type of polyhedral links, which can facilitate the description and characterization of these DNA polyhedra. In this paper, we focus on one type of such polyhedral links introduced in [11], formed from a polyhedron by the method of 'n-branched curve and X-tangled covering'. Now we will describe the construction of such polyhedral links.

To construct such a polyhedral link, a polyhedron P and two types of basic building blocks are needed. One building block is an n-branched curve, the other is a X-tangle, where X maybe a, b, c or d. The two blocks are shown in Fig. 1a and b, respectively. In Fig. 1b, the dot lines of the two leftmost (resp. rightmost) figures denote an edge whose sign is negative (resp. positive). The sign of edge can be shown in Fig. 1c. The construction method as followings:

Firstly, assign a negative sign $(-)$ or a positive sign $(+)$ to every edges of a polyhedron P, then we can obtain a signed graph, denoted P^- or P^+. Secondly, using several X-tangles to cover every edge of P^- or P^+, and replacing the vertex with degree n of the polyhedron P by using n-branched curve. Thirdly, connecting these two building blocks. Then we obtain four classes of polyhedral links, called $L_a(P^-)$, $L_b(P^-)$, $L_c(P^+)$, $L_d(P^+)$. This can be seen in Fig. 2. For example, in $L_a(P^-)$, the edges e_1, e_2, e_3, e_4, e_5, e_6 consist of 2,1,2,1,1,2 a-tangle, respectively. These polyhedral links $L_a(P^-)$, $L_b(P^-)$, $L_c(P^+)$, $L_d(P^+)$ constructed are alternating and minimal. For the more details, see Sect. 4.

In this paper, we will discuss the braid index of the polyhedral links.

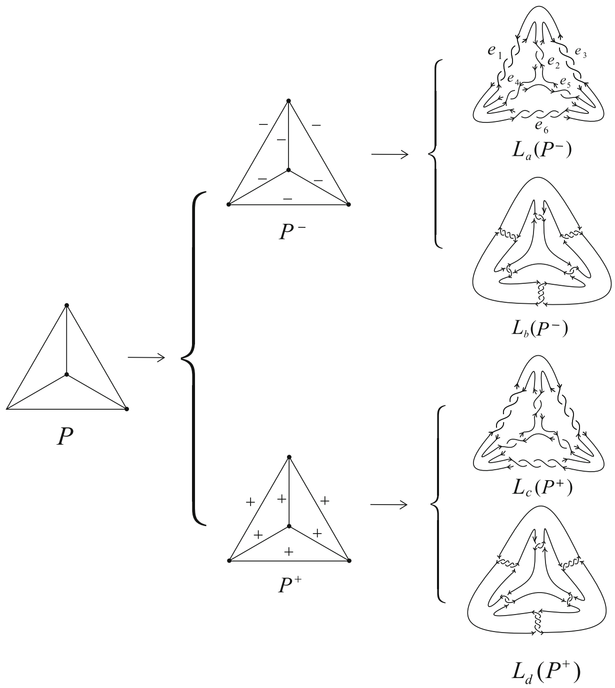


Fig. 2 Four classes of tetrahedral links are formed by the method of 'n-branched *curve* and X-*tangled* covering'

Alexander [12] showed that every link is presented as a closed braid with a finite number of strings. The braid index of an unoriented link is a very powerful invariant, but in general it is difficult to compute. In order to determine the braid index of a link L, one is seeking general lower and upper estimate on the braid index of the link L. Yamada [13] gives an upper bound for the braid index, which is the number of Seifert circles in a given link diagram. Franks and Williams [14], and Morton [15] independently give a lower bound for the braid index in terms of the HOMFLY polynomial. To be more precise, let $P_L(v, z)$ be the HOMFLY polynomial of a link L and max $\deg_v P_L$ (resp. min $\deg_v P_L$) the maximal degree (resp. minimal degree) in v of $P_L(v, z)$. Then a lower bound of the braid index of a link can be shown in following.

$$\frac{1}{2}\,\mathrm{span}_v P_L(v, z) + 1 \le b(L), \tag{1}$$

where $\mathrm{span}_v P_L(v, z) = \max\deg_v P_L(v, z) - \min\deg_v P_L(v, z)$. Here $\max\deg_v P_L(v, z)$ (resp. $\min\deg_v P_L(v, z)$) denotes the maximal degree (resp. minimal degree) in v of polynomial $P_L(v, z)$.

This MFW inequality of many families of links is sharp. For all but five knots ($9_{42}, 9_{49}, 10_{132}, 10_{150}, 10_{156}$) in the standard knot table, up to crossing number 10, the MFW inequality is sharp. Furthermore, this inequality is sharp for torus links and closed positive n-braids with a full twist [14]. On the other hand Murasugi [16] conjectured that for any alternating link, the MFW bound and the braid index coincide, and proved that for any 2-bridge link and any alternating fibred link [16], the MFW bound coincides with the braid index. Next, Murasugi and Przytycki [17] found a counter example for Murasugi's conjecture. In 2004, Nakamura [18] showed that the MFW inequality is sharp for a certain family of closed positive braids.

On the other hand, Ohyama [19], in 1993, proves that if L is a nonsplittable link, $c(L)$ is its crossing number, then

$$b(L) \leq 1 + \frac{c(L)}{2}. \tag{2}$$

In this paper, we obtain that the MFW inequality is sharp for the polyhedral links fabricated by the method of 'n-branched curve and X-tangled covering'. The results demonstrate that the braid index of the polyhedral links is influenced by the number of their crossing numbers. Our research provides a measure to characterize and analyze the structure and complexity of DNA polyhedra. Meanwhile, it can also open a door to the study of entanglement in biopolymers. These are considered significant in view of DNA nanotechnology.

2 Preliminaries

2.1 Some conceptions in knot theory

A *link* is a closed oriented 1-manifold embedded smoothly in the 3-sphere S^3, and a *knot* is a link with one connected component. A *knot or link diagram* is a picture of a projection of a knot onto a plane. A diagram of a link is *minimal* if and only if it has no removed crossings.

The *crossing number* of a link L, denoted $c(L)$, is the least number of crossings that occur in any diagram of the link, that is, the minimal number of double points among all diagrams of a link. It states that:

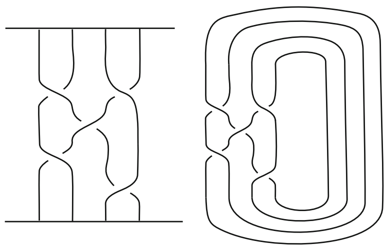
$$c(L) = \min\{c(D) : D \text{ is diagram of } L\}.$$

An $n - stringbraid\ b$ is a set of n arcs embedded in $D^2 \times I$ such that each 2-disc $D^2 \times \{x\}, x \in I$, meets the n arcs in exactly n points, where $n \geq 1$ and $I = [0, 1]$. A *closed n-string braid* $\hat{b}$ is a set of n arcs embedded in $D^2 \times S^1$ such that each disk $D^2 \times \{x\}, x \in S^1$, meets the n arcs in exactly n points. An example is illustrated in Fig. 3.

Alexander [12] showed that every link L in S^3 is represented as a closed braid with a finite number of strings.

The *braid index*, denoted by $b(L)$, is the smallest positive integer n such that L can be represented as a closed n-string braid. Obviously, the braid index is a link type invariant of L, but generally it is not easy to determine the braid index of a link.

Fig. 3 A braid b and its corresponding closed braid $\hat{b}$



2.2 Some conceptions and terminologies in graph theorem

A *graph* G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a non-empty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. Throughout this paper, let p denote the number of vertices of a polyhedron P, q the number of edges.

A *planar* graph is a graph which can be embedded in the plane or the sphere. A planar graph already drawn in the plane without edge intersections is called a *plane graph*. Steinitz, in 1992, indicates that the planar graph of all convex polyhedrons are plane graphs [20].

A graph G is *connected* if it has a path with end vertices u and v, where $u, v \in E(G)$. Otherwise, G is *disconnected*. A *component* of a graph G is a subgraph that is connected and is not contained in any other connected subgraph of G.

3 The Tutte and Homfly polynomials

In order to calculate the braid index of polyhedral links formed by 'n-branched curve and X-tangled covering', we can work with the help of the Tutte polynomial and the Homfly polynomial. Here, we first introduce the conceptions of Tutte and Homfly polynomial.

The Tutte polynomial of a graph was constructed by Tutte in 1954 [21], the polynomial is defined as

$$T(G; x, y) = \sum_{F \subseteq E(G)} (x-1)^{c(F)-1}(y-1)^{|F|-p+c(F)},$$

where F is a subset of $E(G)$, $c(F)$ the number of connected components of $G[F]$, $G[F]$ an induced subgraph of G, and $|F|$ the cardinality of the subset F of $E(G)$.

Another equivalent definition can be given by the following definition.

Definition 3.1 [21] The Tutte polynomial of a graph $G = (V, E)$ is a two-variable polynomial defined as follows:

(1) if $E(G) = \emptyset$, then $T(G; x, y) = 1$.
(2) if e is a bridge, then $T(G; x, y) = xT(G/e, x, y)$.
(3) if e is a loop, then $T(G; x, y) = yT(G - e, x, y)$.

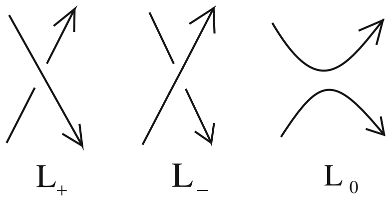


Fig. 4 L_+, L_- and L_0

(4) if e is neither a loop nor a bridge, then $T(G; x, y) = T(G - e, x, y) + T(G/e, x, y)$.

In the above definition, $G - e$ is a graph obtained from G by deleting the edge e, and G/e is obtained from G by contracting e, that is, by deleting e and identifying its two adjacent vertices.

Next, we introduce the definition of Homfly polynomial.

Definition 3.2 [22] The Homfly polynomial of an oriented link L, denoted by $P_L(v, z)$, can be defined by the three following axioms.

(1) $P_L(v, z)$ is invariant under ambient isotopy of L.
(2) If L is the trivial knot then $P_L(v, z) = 1$.
(3) Skein relation: $v^{-1}P_{L_+}(v, z) - vP_{L_-}(v, z) = zP_{L_0}(v, z)$, where L_+, L_- and L_0 are link diagrams which are identical except near one crossing where they are as in Fig. 4 and are called a skein triple.

A link $L(G)$ can be obtained from graph G via the well-known medial construction in knot theory [23]. Thus, there exists a one-to-one correspondence between link diagrams and signed plane graphs via the medial construction [23]. Thereby, for a polyhedral link formed by 'n-branched curve and a-tangled covering', Jeager [24] established a relationship of the Tutte polynomial of a plane graph G and the Homfly polynomials of the link $L(G)$. Let $\widehat{P_a^-}$, $\widehat{P_b^-}$, $\widehat{P_c^+}$, $\widehat{P_d^+}$ be four plane graphs from $L_a(P^-)$, $L_b(P^-)$, $L_c(P^+)$, $L_d(P^+)$ by operation shown in Fig. 1b, respectively. Then graph $\widehat{P_a^-}$, $\widehat{P_c^+}$ is subdivision of the polyhedron P^-, P^+, respectively. $\widehat{P_b^-}$, $\widehat{P_d^+}$ is the graph by adding some parallel edges on P^-, P^+, respectively.

Theorem 3.3 [24] *Let P^- be a connected signed plane graph with p vertices and q edges, and let $L_a(P^-)$ be a link obtained by applying k ($k \geq q$) operations of 'a-tangle covering' to P^-. Then*

$$P_{L_a(P^-)}(v, z) = \left(\frac{v}{z}\right)^{(p+k-q)-1} \left(\frac{z}{v^{-1}}\right)^k T\left(\widehat{P_a^-}, \frac{v^{-1}}{v}, 1 - \frac{-1 + v^2}{z^2}\right). \quad (3)$$

Next, in [11], Zhang et al generalized the results, and respectively gave a relation of the Homfly polynomial of $L_b(P^-)$, $L_c(P^+)$, $L_d(P^+)$ and the Tutte polynomial of the links corresponding plane graphs $\widehat{P_b^-}$, $\widehat{P_c^+}$, $\widehat{P_d^+}$.

Theorem 3.4 [11] *Let P^- and P^+ be two connected signed plane graphs with p vertices and q edges. Let $L_b(P^-)$, $L_c(P^+)$, and $L_d(P^+)$ be four links, respectively obtained by applying k ($k \geq q$) operations of 'b-tangle covering', 'c-tangle covering' and 'd-tangle covering'. Then*

$$P_{L_b(P^-)}(v,z)=\left(-\frac{z}{v^{-1}}\right)^{(p+k-q)-1}\left(-\frac{z}{v}\right)^{k}T\left(\widehat{P_b^-},\frac{v}{v^{-1}},1-\frac{-1+v^{-2}}{z^2}\right)$$
$$P_{L_c(P^+)}(v,z)=\left(-\frac{v^{-1}}{z}\right)^{(p+k-q)-1}\left(-\frac{z}{v}\right)^{k}T\left(\widehat{P_c^+},\frac{v}{v^{-1}},1-\frac{-1+v^{-2}}{z^2}\right)$$
$$P_{L_d(P^+)}(v,z)=\left(\frac{z}{v}\right)^{(p+k-q)-1}\left(\frac{v}{v^{-1}}\right)^{k}T\left(\widehat{P_d^+},1-\frac{v^2-1}{z^2},\frac{v^{-1}}{v}\right).$$

The above Theorems 3.3 and 3.4 can open a door to explore the braid index of the type of polyhedral links.

4 The braid index of the four classes of polyhedral links

In this section, we shall calculate the braid index of the polyhedral links. In combination with formula (1) and (2), for a nonsplittable link, we have

$$\frac{1}{2}\,\mathrm{span}_v P_L(v,z)+1\le b(L)\le 1+\frac{c(L)}{2}. \tag{4}$$

Therefore, if the type of polyhedral links are nonsplittable links, then the formula (4) can be written as follows.

$$\frac{1}{2}\,\mathrm{span}_v P_{L_a(P^-)}(v,z)+1\le b\left(L_a(P^-)\right)\le 1+\frac{c(L_a(P^-))}{2}$$
$$\frac{1}{2}\,\mathrm{span}_v P_{L_b(P^-)}(v,z)+1\le b\left(L_b(P^-)\right)\le 1+\frac{c\left(L_b(P^-)\right)}{2}$$
$$\frac{1}{2}\,\mathrm{span}_v P_{L_c(P^+)}(v,z)+1\le b\left(L_c(P^+)\right)\le 1+\frac{c\left(L_c(P^+)\right)}{2}$$
$$\frac{1}{2}\,\mathrm{span}_v P_{L_d(P^+)}(v,z)+1\le b\left(L_d(P^+)\right)\le 1+\frac{c\left(L_d(P^+)\right)}{2}.$$

Thereby, in order to use the above four formulae, we need to prove that these polyhedral links are nonsplittable links, and calculate their crossing numbers. Here, we first give a lemma.

Lemma 4.1 [25] *An alternating link in a minimal diagram of n crossings has crossing number n.*

From Lemma 4.1, we know that if an alternating polyhedral link discussed above has n crossings, then the link is minimal and has crossing number n.

The definition of splittable link can also be denoted by the knowledge of graph theorem. A link is *split* if it has a link diagram whose plane graph is not connected.

Theorem 4.2 *Let $L_a(P^-)$, $L_b(P^-)$, $L_c(P^+)$, $L_d(P^+)$ be four links constructed from a polyhedron P by the means of ‘ n-branched curve and X-tangled covering’. Then the four links are all nonsplittable and minimal.*

Proof By Lemma 4.1, $L_a(P^-)$, $L_b(P^-)$, $L_c(P^+)$, $L_d(P^+)$ are minimal. The next step is to prove they are nonsplittable links. Let $\mathscr{C}(L_a(P^-))$, $\mathscr{C}(L_b(P^-))$, $\mathscr{C}(L_c(P^+))$, $\mathscr{C}(L_d(P^+))$ be four plane graphs from the polyhedron P. We know that P is connected, so $\mathscr{C}(L_a(P^-))$, $\mathscr{C}(L_b(P^-))$, $\mathscr{C}(L_c(P^+))$, $\mathscr{C}(L_d(P^+))$ must be connected. Therefore, according to the definition of splittable link, the four polyhedral links are nonsplittable links.

In addition, according to the above lemma, we can obtain that the four polyhedral links are minimal. Then, they are nonsplittable and minimal. □

In the following theorem, we will show that the MFW inequality is sharp for the type of polyhedral links.

Theorem 4.3 *Let P^- be a connected signed plane graph with p vertices and q edges. Let $L_a(P^-)$ be the link obtained by applying k 'a-tangle covering' operations to P^-. Then the braid index of polyhedral link $L_a(P^-)$ is* $\dfrac{c\left(L_a(P^-)\right)}{2}+1=k+1$.

Proof According to formula (3),

$$\begin{aligned}
P_{L_a(P^-)}(v,z) &= \left(\frac{v}{z}\right)^{(p+k-q)-1}\left(\frac{z}{v^{-1}}\right)^{k} T\left(\frac{v^{-1}}{v}, 1-\frac{-1+v^2}{z}\right)\\
&= \left(\frac{v}{z}\right)^{(p+k-q)-1}\left(\frac{z}{v^{-1}}\right)^{k}\sum_{F\subseteq E(P^-)}\left(\frac{v^{-1}}{v}-1\right)^{c(F)-1}\\
&\quad\times\left(\frac{1-v^2}{z}\right)^{|F|-(p+k-q)+c(F)}\\
&= v^{(p+k-q)+k-1}z^{k-(p+k-q)+1}\\
&\quad\times\sum_{F\subseteq E(\widehat{P_a^-})}(1-v^2)^{|F|-(p+k-q)+2c(F)-1}v^{-2c(F)+2}z^{-|F|+(p+k-q)-c(F)}.
\end{aligned}$$

If $F=E(T)\cup K$, $K\subseteq E(\widehat{P_a^-})\backslash E(T)$ (T is a spanning tree of $\widehat{P_a^-}$), then $(p+k-q)-1<|F|\le k$ and $c(F)+|F|=1+|F|$.

Others, $c(F)+|F|=(p+k-q)+t$, where t is cyclomatic number of $G|F|$. Also, let $F_1\cup F_2=F$, such that $0\le|F_1|\le(p+k-q)-1$ and $(p+k-q)-1<|F_2|\le k$. Then

$$\begin{aligned}
&P_{L_a(P^-)}(v,z)\\
&= v^{(p+k-q)+k-1}z^{k-(p+k-q)+1}\\
&\quad\times\sum_{F\subseteq E(\widehat{P_a^-})}(1-v^2)^{|F|-(p+k-q)+2c(F)-1}v^{-2c(F)+2}z^{-|F|+(p+k-q)-c(F)}\\
&= v^{(p+k-q)+k-1}z^{k-(p+k-q)+1}\\
&\quad\times\sum_{F_1\subseteq E(\widehat{P_a^-})}(1-v^2)^{|F_1|-(p+k-q)+2c(F_1)-1}v^{-2c(F_1)+2}z^{-|F_1|+(p+k-q)-c(F_1)}
\end{aligned}$$

$$+ v^{(p+k-q)+k-1} z^{k-(p+k-q)+1}$$
$$\times \sum_{F_2 \subseteq E(\widehat{P_a^-})} (1-v^2)^{|F_2|-(p+k-q)+2c(F_2)-1} v^{-2c(F_2)+2} z^{-|F_2|+(p+k-q)-c(F_2)}.$$

Let

$$v^{(p+k-q)+k-1} z^{k-(p+k-q)+1}$$
$$\sum_{F_1 \subseteq E(\widehat{P_a^-})} (1-v^2)^{|F_1|-(p+k-q)+2c(F_1)-1} v^{-2c(F_1)+2} z^{-|F_1|+(p+k-q)-c(F_1)}$$
$$= H(v, z).$$

and

$$v^{(p+k-q)+k-1} z^{k-(p+k-q)+1}$$
$$\sum_{F_2 \subseteq E(\widehat{P_a^-})} (1-v^2)^{|F_2|-(p+k-q)+2c(F_2)-1} v^{-2c(F_2)+2} z^{-|F_2|+(p+k-q)-c(F_2)}$$
$$= K(v, z).$$

Then, $P_{L_a(P^-)}(v, z) = H(v, z) + K(v, z)$.

In the following, we first calculate the max $deg_v P_{L_a(P^-)}(v, z)$.

We know that the max $\deg_v P_{L_a(P^-)}(v, z)$ is the maximum of max $\deg_v H(v, z)$ and max $\deg_v K(v, z)$. So, here we will calculate max $\deg_v H(v, z)$ and max $\deg_v K(v, z)$, respectively.

$$\begin{aligned} \max \deg_v H(v, z) &= (p+k-q)+k-1+2(|F_1|-(p+k-q)+2c(F_1)-1) \\ &\quad -2c(F_1)+2 = (p+k-q)+k-1+t. \end{aligned}$$

and

$$\begin{aligned} \max \deg_v K(v, z) &= (p+k-q)+k-1+2|F_2|-2(p+k-q)+4c(F_2) \\ &\quad -2-2c(F_2)+2 = k-(p+k-q)+1+2|F_2| \\ &= 3k-(p+k-q)+1. \end{aligned}$$

Due to

$$\begin{aligned} \max \deg_v P_{L_a(P^-)}(v, z) &= \max\{\max\ \deg_v\ H(v, z),\ \max\ \deg_v\ K(v, z)\} \\ &= \max\{(p+k-q)+k-1+t, 3k-(p+k-q)+1\} \\ &= 3k-(p+k-q)+1. \end{aligned}$$

In the following, we will calculate the min $\deg_v\ P_{L_a(P^-)}(v, z)$. In a similar way, we have

$$\min\ \deg_v\ P_{L_a(P^-)}(v, z) = k-(p+k-q)+1.$$

From this,

$$\begin{aligned}\operatorname{span}_v P_L(v,z) &= \max\ \deg_v P_{L_a(P^-)}(v,z) - \min\ \deg_v P_{L_a(P^-)}(v,z)\\ &= (3k-(p+k-q)+1)-(k-(p+k-q)+1)\\ &= 2k.\end{aligned}$$

Thus,

$$b\left(L_a(P^-)\right) \geq \frac{\operatorname{span}_v P_L(v,z)}{2}+1 = k+1.$$

In addition, $b\left(L_a(P^-)\right) \leq \dfrac{c\left(L_a(P^-)\right)}{2}+1=\dfrac{2k}{2}+1=k+1.$
Then, we obtain that the braid index of the polyhedral link $L_a(P^-)$ is

$$b\left(L_a(P^-)\right) = \frac{c\left(L_a(P^-)\right)}{2}+1 = k+1.$$

□

In a similar way, we have the following theorem 4.4.

Theorem 4.4 *Let P^- be a connected signed plane graph. Let $L_b(P^-)$ be the link obtained by applying k 'b-tangle covering' operations to P^-. Then the braid index of polyhedral link $L_b(P^-)$ is* $\dfrac{c\left(L_b(P^+)\right)}{2}+1=k+1.$

Since $L_c(P^+)$ (reps. $L_d(P^+)$) is a mirror of $L_a(P^-)$ (reps. $L_b(P^-)$), we can give the following theorem 4.5.

Theorem 4.5 *Let P^+ be a connected signed plane graph. Let $L_c(P^+)$ (reps. $L_d(P^+)$) be the link obtained by applying k 'c-tangle covering' (reps. 'd-tangle covering') operations to P^+. Then the braid index of the polyhedral link $L_c(P^+)$ (reps. $L_d(P^+)$) is* $\dfrac{c\left(L_c(P^+)\right)}{2}+1=k+1.\left(reps. \dfrac{c\left(L_d(P^+)\right)}{2}+1=k+1.\right)$

The above Theorems 4.3, 4.4 and 4.5 demonstrate that the MFW inequality is sharp for the type of 'X-tangle covering' polyhedral links.

5 Conclusions

In this paper, inspired by the braid index of some types of links, we focus on the type of polyhedral links formed from a polyhedron by the method of 'n-branched curve and X-tangled covering', and obtain that the MFW inequality is also sharp for the links. The result demonstrates that the braid index of the polyhedral links is directed by their crossing numbers.

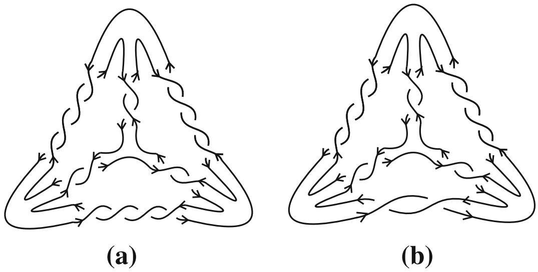


Fig. 5 The link **a** is obtained by applying 9 operations of 'a-tangle covering'. The link **b** is fabricated by applying 8 operations of 'a-tangle covering'

Research on the braid index has provided a new way to detect the complexity of these links. As a result, the type of polyhedral links with greater braid index is more complex. Thereby, the complexity of the four links $L_a(P^-)$, $L_b(P^-)$, $L_c(P^+)$, $L_d(P^+)$ increases with the increase of their crossing numbers. For example, in Fig. 5, the braid index of (a) is 10, the braid index of (b) is 9. So the complexity of the link (a) is more than the link (b).

From mathematics, the research indicates that MFW inequality is sharp for the polyhedral links we introduce above. And from chemistry, the studies of the type of polyhedral links can facilitate the research of the properties of DNA nanotechnology, and can characterize their structural complexity.

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